SS 2016 Dr. Milena Pabiniak
Marc Kegel

Differential Geometry

Homework 2

Mandatory Exercise 1. (10 points)
Let M, N and P be smooth manifolds.

(a) Consider the identity map $f\colon M \to M$. Show that $df\colon TM \to TM$ is also the identity map.

(b) Let $f\colon M \to N$ and $g\colon N \to P$ be two differentiable maps. Show that $g \circ f\colon M \to P$ is also differentiable and that
$$\big(d(g\circ f)\big)_p = (dg)_{f(p)} \circ (df)_p,$$
holds for all $p \in M$.

(c) If $f\colon M \to N$ is a diffeomorphism, then $df\colon TM \to TN$ is also bijective with inverse map given by $d(f^{-1})$.

Mandatory Exercise 2. (10 points)

(a) Give an example of an embedding. And an example of an immersion which is not an embedding.

(b) Show that locally any immersion is an embedding, i.e. if $f\colon M \to N$ is an immersion and $p \in M$, then there exists an open neighborhood W of p in M such that $f|_W$ is an embedding.

(c) Let $f\colon M \to N$ an injective immersion. Show that if M is compact then $f(M)$ is a submanifold of N. Give an counterexample for this fact if M is not compact.

Suggested Exercise 1. (0 points)
Consider the two atlases $\mathcal{A}_1 = \{(\mathbb{R}, \varphi_1)\}$ and $\mathcal{A}_2 = \{(\mathbb{R}, \varphi_2)\}$ on $\mathbb{R}$ given by $\varphi_1(x) = x$ and $\varphi_2(x) = x^3$.

(a) Show that $\varphi_2^{-1} \circ \varphi_1$ is not differentiable and conclude that the two atlases are not equivalent.

(b) The identity map $\mathrm{id}\colon \{(\mathbb{R}, \varphi_1)\} \to \{(\mathbb{R}, \varphi_2)\}$ is not a diffeomorphism.

(c) The map $f\colon \{(\mathbb{R}, \varphi_1)\} \to \{(\mathbb{R}, \varphi_2)\}$ given by $f(x) = x^3$ is a diffeomorphism. Conclude that the two differentiable structures are diffeomorphic.

Suggested Exercise 2. (0 points)
Let $\{(U_\alpha, \varphi_\alpha)\}$ be a differentiable structure on M and consider the maps

$$\begin{aligned}\Phi_\alpha \colon U_\alpha \times \mathbb{R}^n &\longrightarrow TM \\ (x, v) &\longmapsto (d\varphi_\alpha)_x(v) \in T_{\varphi_\alpha(x)}M.\end{aligned}$$

(a) Show that the family $\{(U_\alpha \times \mathbb{R}^n, \Phi_\alpha)\}$ defines a differentiable structure for TM.

(b) Conclude that TM carries the structure of a differentiable manifold. What dimension has TM?

(c) If $f\colon M \to N$ is differentiable, then $df\colon TM \to TN$ is also differentiable.

Suggested Exercise 3. (0 points)
Let M be an n-dimensional differentiable manifold and $p \in M$. Show that the following set can be canonically identified with T_pM (and therefore constitute an alternative geometric definition of the tangent space):
$C_p/_\sim$, where C_p is the set of differentiable curves $c\colon I \to M$ such that $c(0) = p$ and $\sim$ is the equivalence relation defined by

$$c_1 \sim c_2 :\Leftrightarrow \frac{d}{dt}\big(\varphi^{-1} \circ c_1\big)(0) = \frac{d}{dt}\big(\varphi^{-1} \circ c_2\big)(0)$$

for some parametrization $\varphi\colon U \to M$ of a neighborhood of p.

Suggested Exercise 4. (0 points)

(a) Show that the definition of a differentiable map does not depend on the choice of the parametrizations.

(b) A differentiable map is also continuous.

Suggested Exercise 5. (0 points)
The **connected sum** of two topological n-manifolds M and N is the topological manifold $M\#N$ obtained by deleting an open set homeomorphic to a ball on each manifold and gluing the resulting boundaries together by a homeomorphism.

(a) Give examples of this construction.

(b) Show that $M\#N$ is again a topological manifold.

(c) Show that $M\#S^n$ is homeomorphic to M.

(d) Show that $T^2\#\mathbb{R}P^2$ is homeomorphic to $\mathbb{R}P^2\#\mathbb{R}P^2\#\mathbb{R}P^2$.

(e) Is the connected sum of two orientable manifolds again orientable?

Hand in: Monday April 25th
in the exercise session
in Seminar room 2, MI